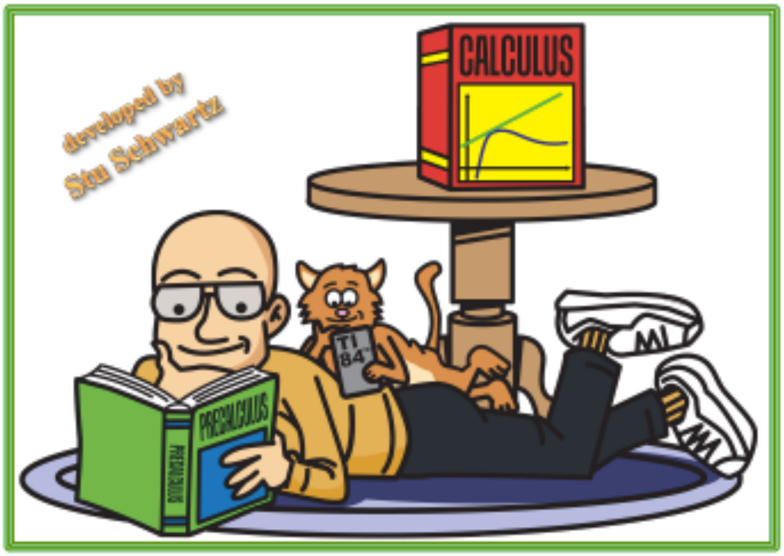
Name: ______________________________

RU Ready for Some Calculus?

A Precalculus Review



Introduction to Calculus
Summer Prep Packet

This packet is for all students entering the Honors Introduction to Calculus class.

Attached, you will find the basic learning targets from Algebra and Pre-Calculus that you are expected to know BEFORE you come to class in the fall. For each topic addressed, this packet contains review examples, properties, definitions, and outline video tutorial links followed by practice problems. This material must be mastered in order for you to be successful in Calculus. Since this material is designed as a review, you are responsible for completing this packet on your own. The packet will be graded to assess the students' efforts to recall this information. Give it your best effort and be sure to **show all your work!!**

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Topics

A. Functions

The lifeblood of precalculus is functions. A **function** is a set of points (x, y) such that for every x, there is one and only one y. In short, in a function, the x-values cannot repeat while the y-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either " $y =$ " or " $f(x) =$ ". In the $f(x)$ notation, we are stating a rule to find y given a value of x.

1. If $f(x) = x^2 - 5x + 8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h) - f(x)}{h}$

a) $f(-6) = (-6)^2 - 5(-6) + 8$

$36 + 30 + 8$

74

b) $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 8$

$\frac{9}{4} - \frac{15}{2} + 8$

$\frac{11}{4}$

c) $\frac{f(x+h) - f(x)}{h}$

$\frac{(x+h)^2 - 5(x+h) + 8 - (x^2 - 5x + 8)}{h}$

$\frac{x^2 + 2xh + h^2 - 5x - 5h + 8 - x^2 + 5x - 8}{5}$

$\frac{h^2 + 2xh - 5h}{h} = \frac{h(h + 2x - 5)}{h} = h + 2x - 5$

Need more? Watch this video for some "live" examples:

Functions do not always use the variable x. In calculus, other variables are used liberally.

2. If $A(r) = \pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1) - A(r)$

$A(3) = 9\pi$

$A(2s) = \pi(2s)^2 = 4\pi s^2$

$A(r+1) - A(r) = \pi(r+1)^2 - \pi r^2$

$\pi(2r+1)$

One concept that comes up in AP calculus is **composition of functions**. The format of a composition of functions is: plug a value into one function, determine an answer, and plug that answer into a second function.

3. If $f(x)=x^2-x+1$ and $g(x)=2x-1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x))\neq g(f(x))$

$g(-1)=2(-1)-1=-3$
$f(-3)=9+3+1=13$

$f(-1)=1+1+1=3$
$g(3)=2(3)-1=5$

$$f(g(x))=f(2x-1)=(2x-1)^2-(2x-1)+1$$
$$=4x^2-4x+1-2x+1+1=4x^2-6x+3$$
$$g(f(x))=g(x^2-x+1)=2(x^2-x+1)-1$$
$$=2x^2-2x+1$$

Need More???

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of x.

4. If $f(x)=\begin{cases}x^2-3, & x\geq 0\\ 2x+1, & x<0\end{cases}$, find a) $f(5)$ b) $f(2)-f(-1)$ c) $f(f(1))$

$f(5)=25-3=22$

$f(2)-f(-1)=1-(-1)=2$

$f(1)=-2,\ f(-2)=-3$

Need more help???

A. Function Assignment

Given $f(x) = 4x - x^2$, find

1. $f(2) =$

2. $f(4) - f(-4) =$

3. $\sqrt{f\left(\frac{3}{2}\right)} =$

4. $f(t) =$

5. $f(x+3) =$

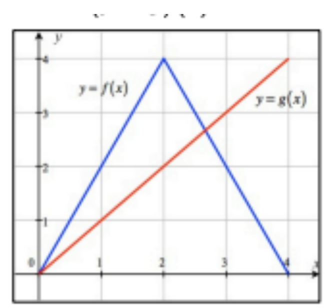
6. $\frac{f(x+h)-f(x)}{h} =$

Given $V(r) = \frac{4}{3}\pi r^3$, find

7. $V(2) =$

8. $V\left(\frac{3}{4}\right) =$

Given $f(x)$ *and* $g(x)$ are given in the graph to the right, find:



9. $g(2) =$

10. $f - g(3) =$

11. $f(g(3)) =$

12. If $f(x) = x^2 - 5x + 3$ *and* $g(x) = 1 - 2x$, then $f(g(x)) =$

Given: If $f(x) = \begin{cases} \sqrt{x+2} - 2, & x \geq 2 \\ x^2 - 1, & 0 \leq x < 2 \\ -x, & x < 0 \end{cases}$, find

13. $f(0) - f(2) =$

14. $\sqrt{5 - f(-4)} =$

15. $f(f(3)) =$

B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of <, ≤, >, ≥ or by using **interval notation**.

Description	Interval notation	Description	Interval notation	Description	Interval notation
$x > a$	(a,∞)	$x \le a$	$(-\infty,a]$	$a \le x < b$	$[a,b)$
$x \ge a$	$[a,\infty)$	$a < x < b$	(a,b) - open interval	$a < x \le b$	$(a,b]$
$x < a$	$(-\infty,a)$	$a \le x \le b$	$[a,b]$ - closed interval	All real numbers	$(-\infty,\infty)$

If a solution is in one interval or the other, interval notation will use the connector $\cup$. So $x \le 2$ or $x > 6$ would be written $(-\infty,2]\cup(6,\infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x < 0$ or $x > 0$ or $(-\infty,0)\cup(0,\infty)$ is best expressed as $x \ne 0$.

More Help?? (Interval Notation)

https://www.youtube.com/watch?v=Ww7xtG2S7IM

The **domain of a function** is the set of allowable x-values. The domain of a function f is $(-\infty,\infty)$ except for values of x which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number. The domain of $a^{p(x)}$ where a is a positive constant and $p(x)$ is a polynomial is $(-\infty,\infty)$.

• Find the domain of the following functions using interval notation:

1. $f(x) = x^2 - 4x + 4$ — $\boxed{(-\infty,\infty)}$
2. $y = \dfrac{6}{x-6}$ — $\boxed{x \ne 6}$
3. $y = \dfrac{2x}{x^2 - 2x - 3}$ — $\boxed{x \ne -1, x \ne 3}$
4. $y = \sqrt{x+5}$ — $\boxed{[-5,\infty)}$
5. $y = \sqrt[3]{x+5}$ — $\boxed{(-\infty,\infty)}$
6. $y = \dfrac{x^2 + 4x + 6}{\sqrt{2x+4}}$ — $\boxed{(-2,\infty)}$

Need some help??? (Domain and Range From a Graph)

https://www.youtube.com/watch?v=fyROLkZc75E

The **range of a function** is the set of allowable y-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible y-value, highest possible y-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y=x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y=\sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y=a^x$ where a is a positive constant is $(0,\infty)$ as constants to powers must be positive.

• Find the range of the following functions using interval notation:

7. $y=1-x^2$

$\boxed{(-\infty,1]}$

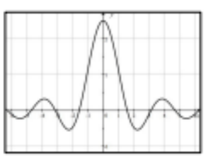
8. $y=\dfrac{1}{x^2}$

$\boxed{(0,\infty)}$

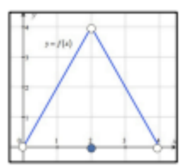
9. $y=\sqrt{x-8}+2$

$\boxed{[2,\infty)}$

• Find the domain and range of the following functions using interval notation.

10. 

Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$

11. 

Domain: $(0,4)$
Range: $[0,4)$

Need some serious help?? (Domain and Range of Rational Functions)
https://www.youtube.com/watch?v=Veq5BBnfMPQ

B. Domain and Range Assignment

Find the domain of the following functions using interval notation:

16. $f(x) = 3$

17. $f(x) = x^3 - x^2 + x$

18. $f(x) = \frac{x^3 - x^2 + x}{x}$

19. $f(x) = \frac{x-4}{x^2-16}$

20. $y = \sqrt{x+3}$

21. $y = 3^x$

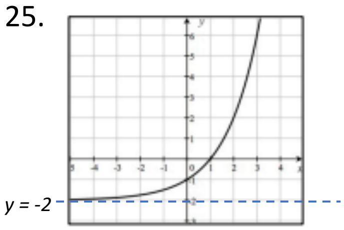
22. $y = \log(x-2)$

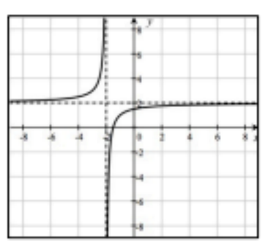
Find the range of the following functions.

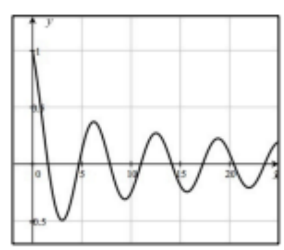
23. $f(x) = x^4 + x^2 - 1$

24. $y = 2^x$

Find the domain and range of the following functions using interval notation.

25. 

26. 

27. 

C. Even and Odd Functions

Functions that are even have the characteristic that for all a, $f(-a)=f(a)$. What this says is that plugging in a positive number a into the function or a negative number $-a$ into the function makes no difference … you will get the same result. Even functions are symmetric to the y-axis.

Functions that are odd have the characteristic that for all a, $f(-a)=-f(a)$. What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the x-axis, it is not a function because it fails the vertical-line test.

1. Of the common functions in section 3, which are even, which are odd, and which are neither?

Even: $y=a$, $y=x^2$, $y=|x|$, $y=\cos x$ Odd: $y=x$, $y=x^3$, $y=\frac{1}{x}$, $y=\sin x$

Neither: $y=\sqrt{x}$, $y=\ln x$, $y=e^x$, $y=e^{-x}$

2. Show that the following functions are even:

a) $f(x)=x^4-x^2+1$

$f(-x)=(-x)^4-(-x)^2+1$
$=x^4-x^2+1=f(x)$

b) $f(x)=\left|\frac{1}{x}\right|$

$f(-x)=\left|\frac{1}{-x}\right|=\left|\frac{1}{x}\right|=f(x)$

c) $f(x)=x^{2/3}$

$f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2$
$=\left(\sqrt[3]{x}\right)^2=f(x)$

3. Show that the following functions are odd:

a) $f(x)=x^3-x$

$f(-x)=(-x)^3+x$
$=x-x^3=-f(x)$

b) $f(x)=\sqrt[3]{x}$

$f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$

c) $f(x)=e^x-e^{-x}$

$f(-x)=e^{-x}-e^x=-\left(e^x-e^{-x}\right)=-f(x)$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither. Justify your answer.

$f(-x)=-x^3-x^2-x-1\neq f(x)$ so f is not even. $-f(x)=-x^3+x^2-x-1\neq f(-x)$ so f is not odd.

Graphs may not be functions and yet have x-axis or y-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$". If the equation does not change after making the following replacements, the graph has these symmetries:

x-axis: y with $-y$ y-axis: x with $-x$ origin: both x with $-x$ and y with $-y$

5. Determine the symmetry for $x^2+xy+y^2=0$.

$x-$axis: $x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $x-$axis
$y-$axis: $(-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $y-$axis
origin: $(-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0$ so symmetric to origin

Need more?? https://www.youtube.com/watch?v=fKyBOLsqRlo

C. Even and Odd Functions – Assignment

Show work to determine if the following functions are even, odd, or neither.

28. $f(x) = 7$

29. $f(x) = 2x^2 - 4x$

30. $f(x) = -3x^3 - 2x$

31. $f(x) = \sqrt{x + 1}$

32. $f(x) = \sqrt{x^2 + 1}$

33. $f(x) = |8x|$

D. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor. The special forms that occur most regularly are:

Common factor: $x^3+x^2+x=x\left(x^2+x+1\right)$
Difference of squares: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=\left(x^n+y^n\right)\left(x^n-y^n\right)$
Perfect squares: $x^2+2xy+y^2=(x+y)^2$
Perfect squares: $x^2-2xy+y^2=(x-y)^2$
Sum of cubes: $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ - Trinomial unfactorable
Difference of cubes: $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$ - Trinomial unfactorable
Grouping: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$

The term "factoring" usually means that coefficients are rational numbers. For instance, x^2-2 could technically be factored as $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$ but since $\sqrt{2}$ is not rational, we say that x^2-2 is not factorable. It is important to know that x^2+y^2 is unfactorable.

• Completely factor the following expressions.

1. $4a^2+2a$

   $\boxed{2a(2a+1)}$

2. $x^2+16x+64$

   $\boxed{(x+8)^2}$

3. $4x^2-64$

   $\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$

   $\boxed{5\left(x^2+y^2\right)(x+y)(x-y)}$

5. $16x^2-8x+1$

   $\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$

   $\boxed{a^2(3a+b)(3a-b)}$

7. $2x^2-40x+200$

   $\boxed{2(x-10)^2}$

8. x^3-8

   $\boxed{(x-2)\left(x^2+2x+4\right)}$

9. $8x^3+27y^3$

   $\boxed{(2x+3y)\left(4x^2-6xy+9y^2\right)}$

10. x^4-11x^2-80

    $\boxed{(x+4)(x-4)\left(x^2+5\right)}$

11. x^4-10x^2+9

    $\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$

    $\boxed{4(3x+4)(3x-4)}$

13. x^3-x^2+3x-3

    $x^2(x-1)+3(x-1)$

    $(x-1)\left(x^2+3\right)$

14. $x^3+5x^2-4x-20$

    $x^2(x+5)-4(x+5)$

    $(x+5)(x-2)(x+2)$

15. $9-\left(x^2+2xy+y^2\right)$

    $9-(x+y)^2$

    $(3+x+y)(3-x-y)$

More help??

https://www.youtube.com/watch?v=KUMhpKGwpCY

D. Special Factorization – Assignment

Completely factor the following expressions.

34. $x^2 - 5x - 24$

35. $x^2 - 81$

36. $x^3 - 25x$

37. $30x - 9x^2 - 25$

38. $33x^8 - 3$

39. $16x^4 - 24x^2y + 9y^2$

40. $9a^4 - a^2b^2$

41. $4x^4 + 7x^2 - 36$

42. $8x^3 - 27$

E. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

Slope: Given two points (x_1, y_1) and (x_2, y_2), the slope of the line passing through the points can be written as: $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Slope intercept form: the equation of a line with slope m and y-intercept b is given by $y = mx + b$.

Point-slope form: the equation of a line passing through the points (x_1, y_1) and slope m is given by $y - y_1 = m(x - x_1)$. While you might have preferred the simplicity of the $y = mx + b$ form in your algebra course, the $y - y_1 = m(x - x_1)$ form is far more useful in calculus.

Intercept form: the equation of a line with x-intercept a and y-intercept b is given by $\frac{x}{a} + \frac{y}{b} = 1$.

General form: $Ax + By + C = 0$ where A, B and C are integers. While your algebra teacher might have required your changing the equation $y - 1 = 2(x - 5)$ to general form $2x - y - 9 = 0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so.

Parallel lines Two distinct lines are parallel if they have the same slope: $m_1 = m_2$.

Normal lines: Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \cdot m_2 = -1$.

Horizontal lines have slope zero. **Vertical lines** have no slope (slope is undefined).

1. Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

   a. $m = -4$, $(1,2)$

   $y - 2 = -4(x - 1) \Rightarrow y = -4x + 6$

   b. $m = \frac{2}{3}$, $(5,1)$

   $y - 1 = \frac{2}{3}(x - 5) \Rightarrow y = \frac{2x}{3} - \frac{7}{3}$

   c. $m = 0$, $\left(-\frac{1}{2}, \frac{3}{4}\right)$

   $y = -\frac{3}{4}$

2. Find the equation of the line in slope-intercept form, passing through the following points.

   a. $(4,5)$ and $(-2,-1)$

   $m = \frac{5+1}{4+2} = 1$

   $y - 5 = x - 4 \Rightarrow y = x + 1$

   b. $(0,-3)$ and $(-5,3)$

   $m = \frac{3+3}{-5-0} = \frac{-6}{5}$

   $y + 3 = \frac{-6}{5}x \Rightarrow y = \frac{-6}{5}x - 3$

   c. $\left(\frac{3}{4}, -1\right)$ and $\left(1, \frac{1}{2}\right)$

   $m = \left(\frac{\frac{1}{2} + 1}{1 - \frac{3}{4}}\right)\left(\frac{4}{4}\right) = \frac{2+4}{4-3} = 6$

   $y - \frac{1}{2} = 6(x - 1) \Rightarrow y = 6x - \frac{11}{2}$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

   a. $(4,7)$, $4x - 2y = 1$

   $y = 2x - \frac{1}{2} \Rightarrow m = 2$

   a) $y - 7 = 2(x - 4)$ b) $y - 7 = \frac{-1}{2}(x - 4)$

   b. $\left(\frac{2}{3}, 1\right)$, $x + 5y = 2$

   $y = \frac{-1}{5}x + 2 \Rightarrow m = \frac{-1}{5}$

   a) $y - 1 = \frac{-1}{5}\left(x - \frac{2}{3}\right)$ b) $y - 1 = 5\left(x - \frac{2}{3}\right)$

More, more, more? Videos to help you with assignment ☺

Equation of Line Given Two Points

https://www.youtube.com/watch?v=4vXqMsvPSv4

Equation of Line Given Point and Slope

https://www.youtube.com/watch?v=AqONrrPhJvk

Equation of a Line Through a Point Parallel or Perpendicular to Another Line

https://www.youtube.com/watch?v=TrONIeOpJHg

E. Linear Functions – Assignment

Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

43. $m = -7, \ (-3, -7)$

44. $m = -\frac{1}{2}, (2, -8)$

45. $m = \frac{2}{3}, \left(-6, \frac{1}{3}\right)$

Find the equation of the line in slope-intercept form passing through the following points.

46. $(-3, 6) and \ (-1, 2)$

47. $\left(-2, \frac{2}{3}\right) and \left(\frac{1}{2}, 1\right)$

Write the equations of the line through the given point a) parallel and b) perpendicular to the given line.

48. $(-6, 2); \ 5\,x + 2y = 7$

49. $(-3, -4); y = \ -2$

50. Find an equation of the line containing (4, -2) and parallel to the line containing (-1, 4) and (2, 3). Put your answer in point-slope form.

F. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The **discriminant** b^2-4ac will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0, \text{ 2 real solutions (if a perfect square, the solutions are rational)}\\ =0, \text{ 1 real solution}\\ <0, \text{ 0 real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1. Solve for x.

a. $x^2+3x+2=0$

$(x+2)(x+1)=0$

$x=-2, x=-1$

b. $x^2-10x+25=0$

$(x-5)^2=0$

$x=5$

c. $x^2-64=0$

$(x-8)(x+8)=0$

$x=8, x=-8$

d. $2x^2+9x=18$

$(2x-3)(x+6)=0$

$x=\frac{3}{2}, x=-6$

e. $12x^2+23x=-10$

$(4x+5)(3x+2)=0$

$x=-\frac{5}{4}, x=-\frac{2}{3}$

f. $48x-64x^2=9$

$(8x-3)^2=0$

$x=\frac{3}{8}$

g. $x^2+5x=2$

$x=\frac{-5\pm\sqrt{25+8}}{2}$

$x=\frac{-5\pm\sqrt{33}}{2}$

h. $8x-3x^2=2$

$x=\frac{8\pm\sqrt{64-24}}{6}$

$x=\frac{8\pm 2\sqrt{10}}{6}=\frac{4\pm\sqrt{10}}{3}$

i. $6x^2+5x+3=0$

$x=\frac{-5\pm\sqrt{25-72}}{12}=\frac{-5\pm\sqrt{-47}}{12}$

No real solutions

j. $x^3-3x^2+3x-9=0$

$x^2(x-3)-3(x-3)=0$

$(x-3)(x^2-3)=0$

$x=3,\ x=\pm\sqrt{3}$

k. $\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}$

$6x\left(\frac{x}{3}-\frac{5}{2}=\frac{-3}{x}\right)$

$2x^2-15x+18=0$

$(2x-3)(x-6)=0$

$x=\frac{3}{2},\ x=6$

l. $x^4-7x^2-8=0$

$(x^2-8)(x^2+1)=0$

$x=\pm\sqrt{8}=\pm 2\sqrt{2}$

Need More???

https://www.youtube.com/watch?v=glQPvmgN5Tw&t=3s

https://www.youtube.com/watch?v=5Nf_2rYbI-w&t=2s

F. Solving Quadratic Equations – Assignment

Solve for x.

51. $x^2 + 7x - 18 = 0$

52. $2x^2 - 72 = 0$

53. $12x^2 - 5x = 2$

54. $81x^2 + 72x + 16 = 0$

55. $x^2 + 10x = 7$

56. $3x - 4x^2 = -5$

57. $x^3 - 5x^2 + 5x - 25 = 0$

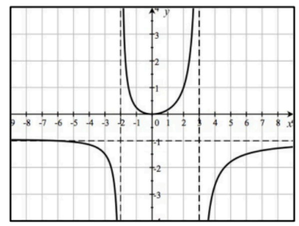
58. $2x^4 - 15x^3 + 18x^2 = 0$

G. Asymptotes and Holes

Rational functions in the form of $y=\dfrac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

Horizontal asymptotes are lines that the graph of the function approaches when x gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of x is in the denominator, the horizontal asymptote is the line $y = 0$. If the highest power of x is both in numerator and denominator, the horizontal asymptote will be the line $y=\dfrac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$. If the highest power of x is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

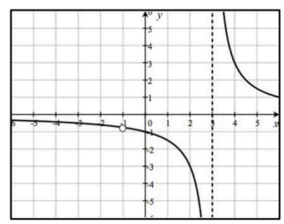
1) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{-x^2}{x^2-x-6}$.



$$y=\frac{-x^2}{x^2-x-6}=\frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes: $x-3=0\Rightarrow x=3$ and $x+2=0\Rightarrow x=-2$
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y=-1$.

This is confirmed by the graph to the right. Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

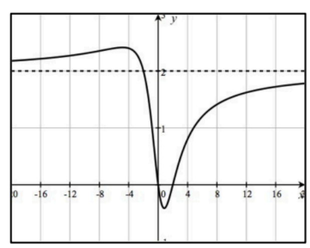


2) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{3x+3}{x^2-2x-3}$.

$$y=\frac{3x+3}{x^2-2x-3}=\frac{3(x+1)}{(x-3)(x+1)}=\frac{3}{x-3}$$

Vertical asymptotes: $x-3=0\Rightarrow x=3$. Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes: Since there the highest power of x is in the denominator, there is a horizontal asymptote at $y = 0$ (the x-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y=\dfrac{2x^2-4x}{x^2+4}$.



$$y=\frac{2x^2-4x}{x^2+4}=\frac{2x(x-2)}{x^2+4}$$

Vertical asymptotes: None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of x is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

Need more???

https://www.educreations.com/lesson/view/pc-notes-2-6-day-1/60375266/?s=eoohta

https://www.educreations.com/lesson/view/pc-notes-2-6-day-2/60376716/?s=0qputx

G. Asymptotes – Assignment

Find any holes first. Then locate all vertical, horizontal, and slant asymptotes.

59. $y = \frac{x-1}{x+5}$

60. $y = \frac{2x+16}{x+8}$

61. $y = \frac{2x^2+6x}{x^2+5x+6}$

62. $y = \frac{x}{x^2-25}$

63. $y = \frac{x^3}{x^2+4}$

64. $y = \frac{2x^2-x+3}{x+1}$

H. Simplifying Complex Fractions

Calculus frequently uses **complex fractions**, which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions:

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$.

However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$.

• Eliminate the complex fractions.

1. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

$$\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}$$

2. $\dfrac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}$$

3. $\dfrac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}$$

4. $\dfrac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}$$

5. $\dfrac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}$$

6. $\dfrac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}$$

7. $\dfrac{x^{-3}+x}{x^{-2}+1}$

$$\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}$$

8. $\dfrac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}$$

9. $\dfrac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

$$\left(\frac{(x-1)^{1/2}-\frac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right]$$

$$\frac{2(x-1)-x}{2(x-1)^{3/2}}=\frac{x-2}{2(x-1)^{3/2}}$$

Videos ~

https://www.youtube.com/watch?v=PpSyx-brMyg

H. Eliminating Complex Fractions - Assignment

Eliminate the complex fractions. Do no leave any negative exponents.

65. $\frac{\frac{5}{8}}{\frac{-2}{3}} =$

66. $\frac{4-\frac{2}{9}}{3+\frac{4}{3}} =$

67. $\frac{2+\frac{7}{2}+\frac{3}{5}}{5-\frac{3}{4}} =$

68. $\frac{x-\frac{1}{x}}{x+\frac{1}{x}} =$

69. $\frac{1+x^{-1}}{1-x^{-2}} =$

70. $\frac{x^{-1}+y^{-1}}{x+y} =$

I. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions: Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* … in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD*. When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1. a. Combine: $\frac{x}{3}-\frac{x}{4}$

$$\text{LCD: } 12 \quad \frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$$
$$\frac{4x-3x}{12}=\frac{x}{12}$$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$$
$$4x-3x=144\Rightarrow x=144$$
$$x=144:\frac{144}{3}-\frac{144}{4}=48-36=12$$

2. a. Combine $x+\frac{6}{x}$

$$\text{LCD: } x \quad x\left(\frac{x}{x}\right)+\frac{6}{x}$$
$$\frac{x^2+6}{x}$$

b. Solve: $x+\frac{6}{x}=5$

$$x(x)+x\left(\frac{6}{x}\right)=5x$$
$$x^2+6=5x\Rightarrow x^2-5x+6=0$$
$$(x-2)(x-3)=0\Rightarrow x=2,\ x=3$$
$$x=2:2+\frac{6}{2}=2+3=5 \quad x=3:3+\frac{6}{3}=3+2=5$$

3. a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

$$\text{LCD: } x(x+2) \quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$$
$$\frac{12x-4x-8}{x(x+2)}$$
$$\frac{8x-8}{x(x+2)}$$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$$
$$12x-4x-8=x^2+2x\Rightarrow x^2-6x+8=0$$
$$(x-2)(x-4)=0\Rightarrow x=2,4$$
$$x=2:\frac{12}{4}-\frac{4}{2}=3-2=1 \quad x=4:\frac{12}{6}-\frac{4}{4}=2-1=1$$

4. a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

$$\text{LCD: } 2(x-3)^2$$
$$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$$
$$\frac{x^2-3x-6}{2(x-3)^2}$$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$$
$$3x(x-3)-18=2(x-3)(x-2)$$
$$3x^2-9x-18=2x^2-10x+12$$
$$x^2+x-30=0\Rightarrow(x+6)(x-5)=0\Rightarrow x=-6,5$$
$$x=-6:\frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27} \quad x=5:\frac{5}{4}-\frac{3}{4}=\frac{3}{6}$$

Video:

https://www.educreations.com/lesson/view/pc-notes-2-7/60375171/?s=6ig4zm

I. Adding Fractions and Solving Fractional Equations – Assignments

71. a. Combine: $\frac{2}{3}-\frac{1}{x}$ b. Solve: $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$

72. a. Combine: $\frac{1}{x-3}+\frac{1}{x+3}$ b. Solve: $\frac{1}{x-3}+\frac{1}{x+3}=\frac{10}{x^2-9}$

73. a. Combine: $\frac{5}{2x}-\frac{5}{3x+15}$ b. Solve: $\frac{5}{2x}-\frac{5}{3(x+5)}=\frac{5}{x}$

J. Exponential and Logarithmic Functions

Calculus spends a great deal of time on exponential functions in the form of b^x. Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$. So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$.

If the base of a log statement is not specified, it is defined to be 10. When we asked for log 100, we are solving the equation: $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0,\infty)$ and range $(-\infty,\infty)$. In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$. Students should know that the value of $e = 2.71828\ldots$

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a \cdot b)$ ii. $\log a - \log b = \log\left(\frac{a}{b}\right)$ iii. $\log a^b = b \log a$

1. Find a. $\log_4 8$

$$\log_4 8 = x \quad 4^x = 8 \Rightarrow 2^{2x} = 2^3 \quad x = \frac{3}{2}$$

b. $\ln\sqrt{e}$

$$\ln\sqrt{e} = x \quad e^x = e^{1/2} \quad x = \frac{1}{2}$$

c. $10^{\log 4}$

$\log 4 = x$

$10^x = 4$ so $10^{\log 4} = 4$

10 to a power and log are inverses

d. $\log 2 + \log 50$

$$\log(2 \cdot 50) = \log 100 \quad 2$$

e. $\log_4 192 - \log_4 3$

$$\log_4\left(\frac{192}{3}\right) \quad \log_4 64 = 3$$

f. $\ln\sqrt[5]{e^3}$

$$\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}$$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \frac{1}{2}$

$$x^2 - x + 3 = 9^{1/2} \quad x(x-1) = 0 \quad x = 0, x = 1$$

b. $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$$\log_{36} x(x-1) = \frac{1}{2} \quad x(x-1) = 36^{1/2} = 6 \quad x^2 - x - 6 = 0 \quad (x-3)(x+2) = 0$$

Only $x = 3$ is in the domain

c. $\ln x - \ln(x-1) = 1$

$$\ln\left(\frac{x}{x-1}\right) = 1 \quad \frac{x}{x-1} = e \Rightarrow x = ex - e \quad x = \frac{e}{e-1}$$

d. $5^x = 20$

$$\log\left(5^x\right) = \log 20 \quad x\log 5 = \log 20 \quad x = \frac{\log 20}{\log 5} \text{ or } x = \frac{\ln 20}{\ln 5}$$

e. $e^{-2x} = 5$

$$\ln e^{-2x} = \ln 5 \quad -2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2}$$

f. $2^x = 3^{x-1}$

$$\log\left(2^x\right) = \log\left(3^{x-1}\right) \quad x\log 2 = (x-1)\log 3 \quad x\log 2 = x\log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2}$$

Video Solving Log Equations ~

https://www.youtube.com/watch?v=iBZ0ArS0ZVY&t=4s

Video Properties of Exponents

https://www.youtube.com/watch?v=7ryfCvDzIKY&t=117s

J. Exponential and Logarithmic Functions – Assignment

Simplify each. You should be able to do so without a calculator!

74. $\log_2 \frac{1}{4}$

75. $\log_8 4$

76. e^{ln12}

77. $\log_{\frac{1}{3}} \frac{4}{3} - \log_{\frac{1}{3}} 12$

78. $\log_3 \left(\sqrt{3}\right)^5$

79. $\log_2 \frac{2}{3} + \log_2 \frac{3}{32}$

Solve. Be sure to check your solution in the original equation.

80. $\log_5(3x - 8) = 2$

81. $\log(x - 3) + \log 5 = 2$

82. $\log_5(x + 3) - \log_5 x = 2$

83. $3^{x-2} = 18$

84. $e^{3x+1} = 10$